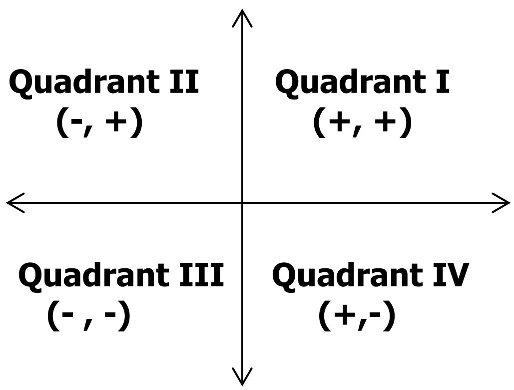
P.1 The Cartesian Plane

The Cartesian Plane

The Cartesian Plane (also called the rectangular coordinate system) is the plane that allows you to represent ordered pairs of real numbers by points. It is names after the French mathematician Rene Descartes (1596 – 1650).



Each point in the plane corresponds to an *ordered pair* (x, y) of real numbers *x* and *y*, called the *coordinates* of the point.

The beauty of the rectangular coordinate system is that it allows you to see relationships between two variables. This is a valuable skill in most scientific and business-related field.

Sketching a Scatter Plot

The amounts *A* (in millions of dollars) spent on baseballs in the United States for the years 1984 to 1993 are given in the table, where *t* represents the year. Sketch a scatter plot of the data.

t	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003
A	616	681	773	830	766	769	776	711	678	685

You can sketch a scatterplot with a graphing calculator:

1. Go to STAT → 1:EDIT → Enter
2. Clear any lists that are already stored in the calculator.
   To do this, highlight the list name (L_1, L_2, etc.) and hit CLEAR and ENTER.
3. Enter the *t* – values into L_1 and the *A* – values into L_2.
4. Set the appropriate WINDOW.
   a. Hit WINDOW.
   b. Set the Xmin just below your smallest *t* – value. Usually with a real- life problem you want to use Xmin = 0.
   c. Set the Xmax just above your largest *t* – value.
   d. Set the Ymin just below your smallest *A* – value. Usually with a real- life problem you want to use Ymin = 0.
5. Turn the Scatterplot ON.
   a. Hit 2^{nd} → Y=.

b. Select 1:Plot 1…Off
c. Hit ENTER so the Plot turns On
d. Make sure the scatterplot type of graph is chosen. (It is the first TYPE option)
e. Make sure your Xlist is coming from L_1 and your Ylist is coming from L_2.
f. Change the "Mark" to the box (instead of the dot).

6. Select GRAPH to see the scatterplot.

Analyzing the data in a scatterplot

To analyze a scatterplot the calculator will calculate various Regressions. Suppose you think the data in the table above looks linear. To test the strength of your prediction you can run a Linear Regression. To do this, follow these steps:

1. Quit out of the scatterplot back to the home screen.
2. Select STAT→ CALC → 4:LinReg(ax + b) → L_1, L_2, Y_1 → ENTER.
3. What you see on the home screen are the the values for *a* and *b*, the slope and y-intercept of the regression equation.
4. If you hit GRAPH you will see the equation overlaid on your scatterplot.

Note: These same steps will get you QuadReg, CubicReg, etc.

Distance & Midpoint Formula and Pythagorean Theorem

Distance: $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Midpoint: $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Pythagorean Theorem: $a^2 + b^2 = c^2$

Standard Form of a Circle

Standard Form: $(x-h)^2 + (y-k)^2 = r^2$ → Center: (h,k) Radius: r

Example

The point (1,2) lies on a circle whose center is at (-3,4). Find an equation for the circle.

Solution

The radius *r* of the circle is the distance between (1,2) and (-3,4).

$$\begin{aligned} r &= \sqrt{(-3-1)^2 + (4-2)^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned}$$

Thus, the center of the circle is (h,k) = (-3,4) and the radius is $r = \sqrt{20}$, and you can write the standard form of the equation of the circle as follows.

$$(x-h)^2+(y-k)^2=r^2$$
$$(x-(-3))^2+(y-4)^2=\left(\sqrt{20}\right)^2$$
$$(x+3)^2+(y-4)^2=20$$

Translating Points in the Plane

A triangle has vertices at the points (-1,2), (2,4), and (3,- 4). Shift the triangle 2 units to the right and 3 units down and find the vertices of the shifted triangle.

(-1, 2) → (-1 +2, 2 – 3) = (1, -1)

(2, 4) → (2 + 2, 4 – 3) = (4, 1)

(3, -4) → (3 + 2, -4 – 3) = (5, -7)

Practice Examples:

1.) Sketch a scatterplot of the data given in the table. The number *y* of Save Mor stores for each year from 1992 through 2000 is given in the table.

t	1992	1993	1994	1995	1996	1997	1998	1999	2000
y	859	980	1114	1259	1399	1568	1714	1850	1953

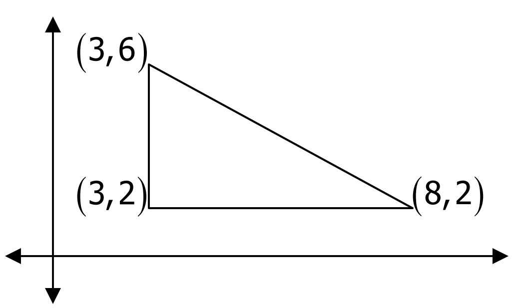
Is this data more Linear or more Quadratic? Why?

2.) A triangle with vertices at (-2,-4), (-1,-1), and (2,-3) is shifted 3 units left and 5 units up.

a. Find the coordinates of the vertices of the new triangle.

b. Sketch the original triangle and the new triangle on the same coordinate grid.

3.) Given the triangle below, use the Distance Formula to find the length of each side of the triangle, and then show that these lengths satisfy the Pythagorean Theorem.

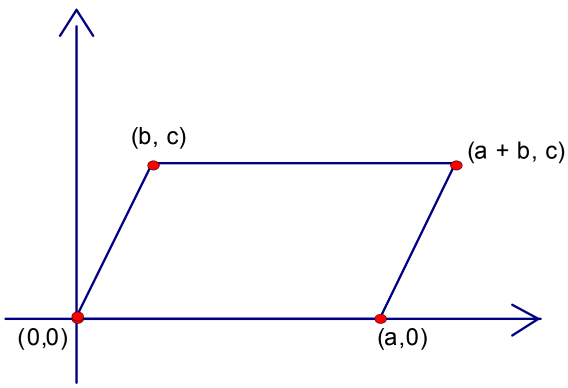


4.) Find the distance and midpoint between $\left(\frac{1}{2},1\right)$ and $\left(-\frac{5}{2},\frac{4}{3}\right)$.

5.) Find the standard form equation of the circle passing through (4,5) with a center at (-1,1).

6.) A plane flies in a straight line to a city that is 80 kilometers west and 120 kilometers south of the point of departure. How far does it fly?

7.) Prove that the diagonals of a parallelogram bisect each other.



P.2 Graphs and Graphing Utilities

The Graph of an Equation

Solution point: For an equation in variables x and y, a point (a,b) is a solution point if the substitution of $x = a$ and $y = b$ satisfies the equation.

Intercepts: the points at which a graph touches or crosses and axis.

Using a Graphing Utility to graph and equation

1. Solve the equation for y.
2. Enter the equation into the Y= screen.
3. Determine a **viewing rectangle** that shows all important attributes of the graph. The standard viewing rectangle on most graphing calculators ranges between –10 and 10 for both the x- and y- values.
4. Hit the GRAPH key to see the graph.

Determining an appropriate Viewing Rectangle

In general you can use six values to determine the viewing rectangle for an equation:

1. the minimum *x*- value
2. the maximum *x*- value
3. the *x*- scale
4. the minimum *y*- value
5. the maximum *y*- value
6. the *y*- scale

Changing the *x*- or *y*- scale: Only necessary when you want the axes to increase in increments greater or less than one. This will allow the calculator to graph an equation quicker.

Sketching a Circle with a Graphing Calculator

A circle can be tricky to graph because it is not a function. The graphing calculator can only graph functions, so you must enter circles as two separate functions.

Example: Use a graphing utility to graph $x^2 + y^2 = 25$.

Solution: This is a circle with center at (0,0) and a radius of 5.

- To graph this equation, begin by solving the equation for *y.*

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25 - x^2}$$

- The graph of $y = \sqrt{25 - x^2}$ is the upper semicircle.
- The graph of $y = -\sqrt{25 - x^2}$ is the lower semicircle.
- To get an graph that "looks" like a circle you must use the SQUARE setting on the calculator. To change the calculator to this setting, hit ZOOM and choose ZSQUARE.

INSERT GRAPH

Practice Exercises

Graph each equation. Approximate any *x*- and *y*- intercepts of the graph.

1. $y = x^2 - 3x + 2$
2. $y = x\sqrt{x - 8}$
3. $y = \dfrac{3x}{x - 2}$
4. $y = x^3 - 3x + 1$
5. $y = 9 - x^2$
6. $y = \sqrt{2x - 5}$

7. $y=\frac{1}{2}(x-4)(x+2)$ 8. $y=\sqrt[3]{x}-2$

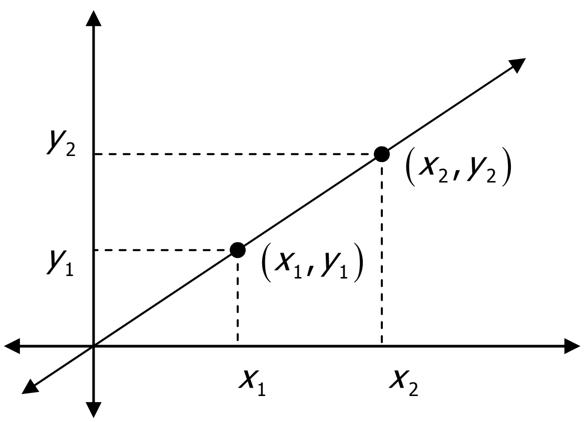
9. A rectangle of length x and width w has a perimeter of 16 meters.
   a) Draw a rectangle that gives a visual representation of the problem.

   b) Show that $w = 8 - x$ is the width of the rectangle.

   c) Use your graphing calculator to graph the equation.
   d) From the graph in part (c), estimate the dimensions of the rectangle that yield a maximum area.

P.3 Lines in the Plane

Slope: the number of units a line rises or falls vertically for each unit of horizontal chance from left to right.



Slope = $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\text{change in } y}{\text{change in } x}$

where $x_1 \neq x_2$

** Vertical Lines have NO SLOPE
** Horizontal Lines have ZERO SLOPE

Equations of Lines

Slope – Intercept Form: $y = mx + b$
Point – Slope Form: $y - y_1 = m(x - x_1)$
Standard (General) Form: $Ax + Bx + C = 0$, A & $B \neq 0$
Vertical Line: $x = a$
Horizontal Line: $y = b$

Parallel and Perpendicular Lines

Parallel Lines → Same Slope *Perpendicular Lines* → Opposite Reciprocal Slope

Example: Find the equation of the line that passes through (3, -2) and has a slope of 2.

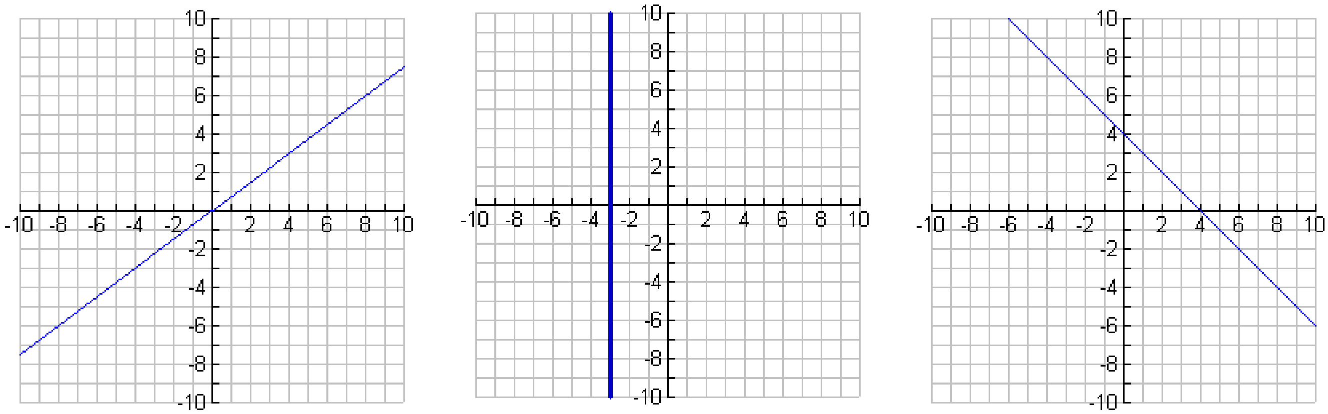
Solution: $y - y_1 = m(x - x_1)$

$$y-(-2)=2(x-3)$$
$$y+2=2x-6$$
$$y=2x-8$$

Example : Sketch the graph of the following equations:

(a) $y=\frac{3}{4}x$ (b) $x=-3$ (c) $x+y=4$

Solution:



Example: Find an equation of a line that passes through (3,2) and is parallel to the line $x-2y=4$.

Solution:

The given line has a slope of $\frac{1}{2}$. Using point-slope form, you can find the equation of the line.

$$y-2=\frac{1}{2}(x-3)$$
$$y-2=\frac{1}{2}x-\frac{3}{2}$$
$$y=\frac{1}{2}x-\frac{3}{2}+2$$
$$y=\frac{1}{2}x+\frac{1}{2}$$

Example: Find an equation of a line that passes through (3,2) and is perpendicular to $x - 2y = 4$.

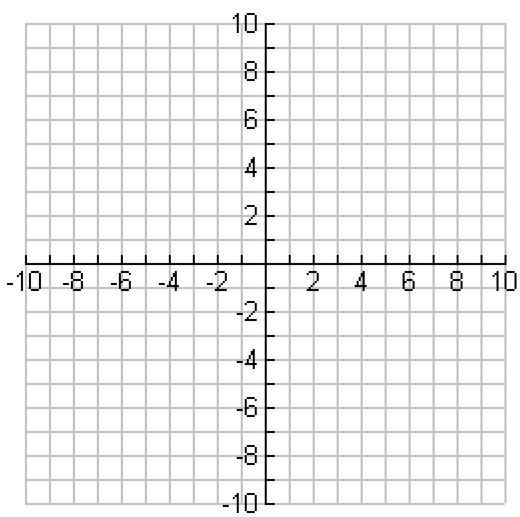
Solution:

The given line has a slope of $\frac{1}{2}$. The perpendicular line will have a slope of -2. Using point-slope form, you can find the equation of the line.

$$y - 2 = -2(x - 3)$$
$$y - 2 = -2x + 6$$
$$y = -2x + 6 + 2$$
$$y = -2x + 8$$

Practice Exercises

1. Sketch the line with a slope of 2 passing through the point (3, -1).



2. Write the equation of the line you graphed in #1 is the following forms:
   (a) Point – Slope Form: ___________________
   (b) Slope – Intercept Form: _________________
   (c) Standard (General) Form: _________________

3. Write the equation of the line that passes through (2,0) and (0,-8) in standard form.

4. Given L_1 and L_2 passing through the following points:

$$L_1:(0,2),(4,-2) \quad L_2:\left(-\frac{1}{2},3\right),\left(\frac{5}{2},6\right)$$

Are L_1 and L_2 parallel, perpendicular, or neither?

5. The length and width of a rectangular garden are 16 feet and 10 feet, respectively.

A walkway of width x surrounds the garden.

(a) Write the outside perimeter of the walkway in terms of x.

(b) Use a graphing calculator to graph the equation.

(c) Find the slope of the graph:__________

(d) What is the meaning of the slope in the context of the problem?

P.4 Solving Equations Algebraically & Graphically

Equation: a statement that two expressions are equal

Solutions: all values of x for which the equation is true

Identity: an equation that is true for *every* real number in the domain of the variable.
Ex. $x^2-9=(x-3)(x+3)$

Conditional Equation: an equation that is true of just some (or even none) of the real numbers in the domain of the variable.

Extraneous Solutions: solutions that do not satisfy the original equation.
→ To avoid: always check your solutions!

Example: Solve: $\frac{2}{x-3}=\frac{4}{x-3}-\frac{6x}{x^2-9}$

Solution:

$$\frac{x+3}{x+3}\bullet\frac{2}{x-3}=\frac{x+3}{x+3}\bullet\frac{4}{x-3}-\frac{6x}{x^2-9}$$

$$\frac{2x+6}{x^2-9}=\frac{4x+12}{x^2-9}-\frac{6x}{x^2-9}$$

$$\frac{2x+6}{x^2-9}=\frac{-2x+12}{x^2-9}$$

$$2x+6=-2x+12$$

$$4x=6$$

$$x=\frac{3}{2}$$

<u>Check your solution:</u>

$$\frac{2}{\left(\frac{3}{2}\right)-3}=\frac{4}{\left(\frac{3}{2}\right)-3}-\frac{6\left(\frac{3}{2}\right)}{\left(\frac{3}{2}\right)^2-9}$$

$$\frac{2}{\left(-\frac{3}{2}\right)}=\frac{4}{\left(-\frac{3}{2}\right)}-\frac{9}{\frac{27}{4}}$$

$$-4=\frac{-8}{3}-\frac{36}{27}$$

$$-4=\frac{-72}{27}-\frac{36}{27}$$

$$-4=-\frac{108}{27}$$

Solution works! ⟶ $-4=-4$

Intercepts & Solutions

x – intercept: the point (a, 0) that is a solutions point of the equation

y – intercept: the point (0, b) that is a solutions point of the equation

To find intercepts:

x – intercept: set $y = 0$ and solve
y – intercept: set $x = 0$ and solve

Example: Find the x and y intercepts of $2x - 5y = 10$.

Solution:

x – intercept → $2x - 5(0) = 10$
$2x = 10$
$x = 5$
(5,0)

y – intercept → $2(0) - 5y = 10$
$-5y = 10$
$y = -2$
(0, -2)

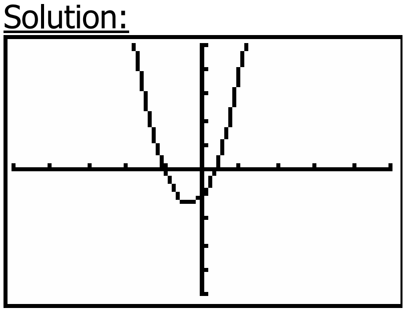
Finding Solutions Graphically

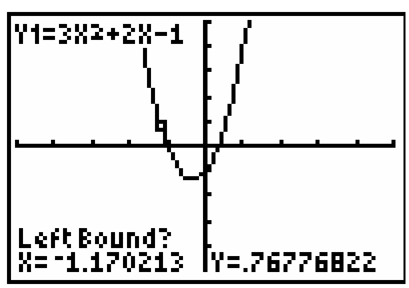
Graphical Approximation:

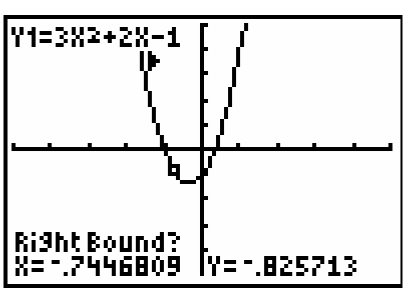
1. Write the original equation in standard form and set it equal to zero. $[f(x) = 0]$
2. Input $f(x)$ into y on calculator.
3. Choose the appropriate window.
4. To find solutions, find the x – intercept(s):
   a. 2nd CALC → ZERO
   b. Choose LEFT BOUND → ENTER
   c. Choose RIGHT BOUND → ENTER
   d. Guess → ENTER

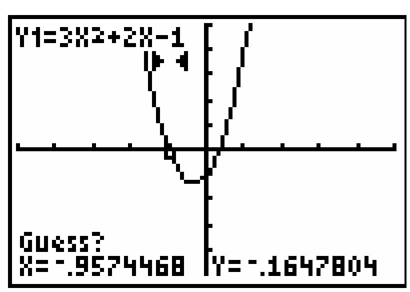
Example: Approximate the solutions of $3x^2 - 2x - 1 = 0$.

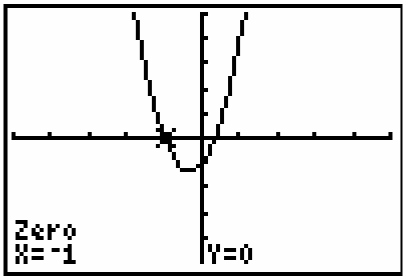
Solution:



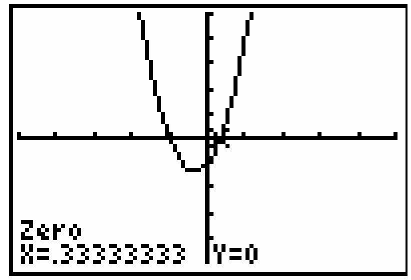
Left Bound: 

Right Bound: 

Guess: 

Solution (zero): 

Repeat steps to find the other zero:



Point(s) of Intersection: An ordered pair(s) that is a solution of two different equations.

Example: Find all points of intersection between the following equations:

$$y = x + 3$$
$$y = x^2 - 3x - 4$$

Solution:

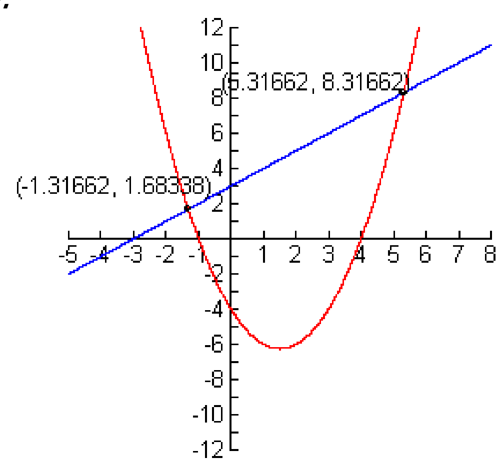
$$x + 3 = x^2 - 3x - 4$$
$$0 = x^2 - 4x - 7$$

This expression is not factorable so you must either use the quadratic formula or the graphing calculator to approximate the solution(s).

Quad Formula:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-7)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{(16) + (28)}}{2}$$
$$= \frac{4 \pm \sqrt{44}}{2} = 2 \pm \sqrt{11} \approx 5.317, -1.317$$
$$(5.317, 8.317); (-1.317, 1.683)$$

Graphing:



Polynomial Equations

Classification of polynomials b y their degree:

Degree	*Name*	*Example*
First degree	Linear Equation	$3x+1=6$
Second degree	Quadratic Equation	$x^2-3x+8=0$
Third degree	Cubic Equation	$2x^3-4x+9=0$
Fourth degree	Quartic Equation	$x^4-3x^2+2x+8=0$
Fifth degree	Quintic Equation	$x^5-x^4+3x^2+5=0$

It is usually harder to solve equations as the degree increases.

Solving Quadratic Equations Algebraically

1. Factoring: If $ab = 0$, then $a = 0$ or $b = 0$.

Example: Solve $x^2-3x+2=0$.

Solution: Factor into: $(x-2)(x-1)=0$

$x=2$ or $x=1$

Special Factoring Patterns

Difference of Squares: $u^2-v^2=(u-v)(u+v)$

Sum of Squares: $u^2+v^2=(u-vi)(u+vi)$

Sum of Cubes: $u^3-v^3=(u-v)(u^2+uv+v^2)$

Difference of Cubes: $u^3+v^3=(u+v)(u^2-uv+v^2)$

2. Square Root Principle: If $u^2=c$, where $c > 0$, then $u = \pm\sqrt{c}$

Example: Solve $(x-4)^2=25$.

Solution: Square Root → $\sqrt{(x-4)^2}=\pm\sqrt{25}$

$(x-4)=\pm 5$

$x=4\pm 5$

$x=9$ or $x=-1$

3. *Completing the Square:* If $x^2 + bx + c = 0$, then

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

$$\left(x + \frac{b}{2}\right)^2 = c + \frac{b^2}{4}$$

Example: Solve: $x^2 + 4x = 5$.

Solution: Complete the Square → $x^2 + 4x + \left(\frac{4}{2}\right)^2 = 5 + \left(\frac{4}{2}\right)^2$

$$x^2 + 4x + 4 = 5 + 4$$

$$(x+2)^2 = 9$$

$$\sqrt{(x+2)^2} = \pm\sqrt{9}$$

$$x + 2 = \pm 3$$

$$x = -5 \text{ or } x = 1$$

4. *Quadratic Formula:* If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example: Solve: $3x^2 + 5x + 1 = 0$

Solution: Use Quadratic Formula: $x = \frac{-5 \pm \sqrt{5^2 - 4(3)(1)}}{2(3)}$

$$= \frac{-5 \pm \sqrt{25-12}}{6} \quad --> \quad x = \frac{-5 \pm \sqrt{13}}{6}$$

The methods listed above can sometimes be used to polynomial equations of higher degree!

Example: Solve the equation $x^4 - x^2 - 6 = 0$

Solution: Factor → $(x^2 - 3)(x^2 - 2) = 0$.

So, $x^2 - 3 = 0$ or $x^2 - 2 = 0$

And $x = \pm\sqrt{3}, \pm\sqrt{2}$

Example: Solve the equation $x^3 - 4x^2 - 2x + 8 = 0$.

Always check your solutions in the original equation. They might be extraneous.

Solution: Factor by grouping → $x^2(x-4)-2(x-4)=0$

$$(x^2-2)(x-4)=0$$

$$x=\pm\sqrt{2} \text{ or } x=4$$

Other Types of Equations

Radical Equations

Example: Solve: $\sqrt{2x-4}+x=4$. → Domain: $x\geq 2$

Solution:

$$\sqrt{2x-4}=4-x$$ → Isolate the radical expression

$$\left(\sqrt{2x-4}\right)^2=(4-x)^2$$

→ Square both sides

$$2x-4=16-8x+x^2$$

$$0=x^2-10x+20$$

→ Set equal to zero

Now, solve using the Quadratic formula → $x=5\pm\sqrt{5}$ → Both work in original ☺

Example: Find all real solutions: $\sqrt{2x+4}-\sqrt{x-2}=1$

Solution:

$$\sqrt{2x+4}=1+\sqrt{x-2}$$

$$\left(\sqrt{2x+4}\right)^2=\left(1+\sqrt{x-2}\right)^2$$

$$2x+4=1+2\sqrt{x-2}+x-2$$

$$x+5=2\sqrt{x-2}$$

$$\frac{x+5}{2}=\sqrt{x-2}$$

$$\left(\frac{x+5}{2}\right)^2=\left(\sqrt{x-2}\right)^2$$

$$\frac{x^2+10x+25}{4}=x-2$$

$$x^2+10x+25=4x-8$$

$$x^2+6x+33=0$$

Now, use Quadratic Formula to solve → x = no real solution!

Equations with Fractions

Example: Find all real solutions: $\frac{3}{x}=\frac{2}{x-2}-1$

Solution:

$$\frac{3}{x}=\frac{2}{x-2}-\frac{1(x-2)}{1(x-2)}$$

$$\frac{3}{x}=\frac{2-x+2}{x-2}$$

$$\frac{3}{x}=\frac{4-x}{x-2}$$

Then use cross multiplication: $3x-6=4x-x^2$

$$0=x^2-x-6$$

$$0=(x-3)(x+2)$$

$x=3$ or $x=-2$

Both solutions work!

Absolute Value Equations

Example: Find all solutions: $|x^2-2x|=-2x+4$

Solution: $x^2-2x=-2x+4$ or $-(x^2-2x)=-2x+4$

$$x^2=4$$

$$x=\pm 2$$

$$-x^2+2x=-2x+4$$

$$x^2-4x+4=0$$

$$(x-2)^2=0$$

$$x=2$$

Only solutions are $x=\pm 2$

Practice Exercises

Determine whether the equation is an identity or a conditional equation:

1. $3(x-2)=3x-6$
2. $x^2+6x-8=x^2+2(3x-5)$

Solve each equation and verify each solution.

3. $2(x-5)-7=3(x+5)$
4. $\frac{3x}{2}+\frac{1}{3}=x-\frac{3}{5}$
5. $x^2-6x-10=0$
6. $\frac{5x}{2}+\frac{1}{4}(x-2)=10$

7. $\dfrac{x+1}{4}+\dfrac{x+1}{x+2}=0$

8. $\sqrt{x-8}-4=0$

9. $\sqrt{x-1}+3x=2$

10. $|3x-2|=10$

11. $|x|=x^2+x-3$

12. $x^4-3x^2-12=6$

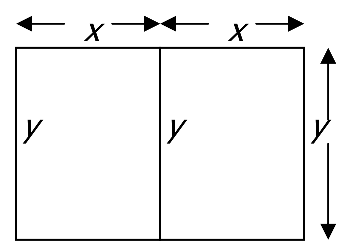
13. $(x-6)^{2/3}=16$

14. Find any points of intersection *algebraically*: $y=\dfrac{2}{3}x-1$

$y-\dfrac{5}{2}x=10$

15. Solve for *h*: $S=\pi r\sqrt{r^2+h^2}$

16. A farmer has 120 meters of fencing to enclose two adjacent corrals (see figure).



a) Write the area of the enclosed region as a function of *x*.

b) Use the graphing calculator to approximate the dimensions that will produce a maximum area.

c) Approximate the dimensions such that the enclosed area will be at least 360 square meters.

d) Find the required dimensions of part (c) analytically.

P. 5 Solving Inequalities Algebraically and Graphically

Properties of Inequalities

Let *a, b,* and *c* be real numbers.

1. *Transitive Property*
   If $a < b$ and $b < c$ then $a < c$
2. *Addition Property of Inequalities*
   If $a < b$ and $c < d$ then $a + c < b + d$
3. *Addition of a constant*
   If $a < b$ then $a + c < b + c$
4. *Multiplying by a constant*
   For $c > 0$, if $a < b$, then $ac < bc$
   For $c < 0$, if $a < b$, then $ac > bc$

Solving Linear Inequalities

Example: Solve the inequality: $6x - 8 \le 2x + 6$

Solution: $6x - 8 \le 2x + 12$

$4x \le 20$

$x \le 5$

> Note:
> Graphically, this is the set of all real numbers less than or equal to 5. The **interval notation** for this solution set is $(-\infty, 5]$. The bracket notation is used to indicate that 5 is included in the solution set. If the solution set had been $x < 5$, the solution set would have been $(-\infty, 5)$ to indicate that 5 is not included in the solution set.

Example: Solve the inequality: $-3 < 8x - 3 \le 13$

Solution: $-3 < 8x - 3 \le 13$

$0 < 8x \le 16$

$0 < x \le 2$

The interval notation for this solution set is $(0, 2]$

Absolute Value Inequalities

Example: $|x - 8| \le 2$

Solution: This is a "less than" situation. It becomes a compound inequality when you solve it:

$|x - 8| \le 2$

$-2 \le x - 8 \le 2$

$6 \le x \le 10$

The solution to this inequality is any number greater than 6 AND less than or equal to 10.

Example: $|x-8|>2$

Solution: This is a "greater than" situation. It becomes two separate inequalities connected by the word OR.

$$|x-8|>2$$
$$x-8>2 \quad \text{OR} \quad x-8<-2$$
$$x>10 \quad \text{OR} \quad x<-6$$

Polynomial Inequalities

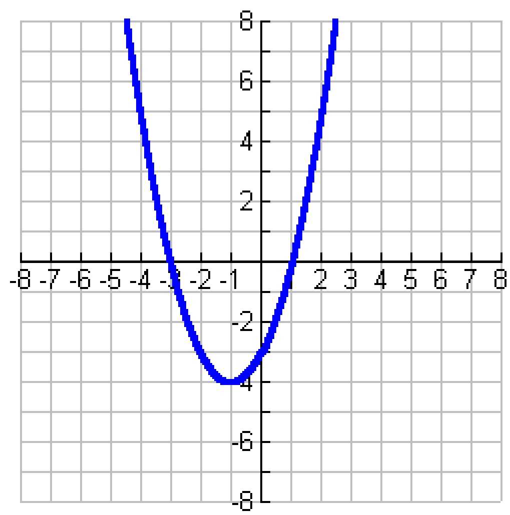
A polynomial can only change signs at its zeros (the x-values that make the polynomial equal to zero). Between two consecutive zeros a polynomial must be entirely positive or entirely negative. Therefore, the real zeros of a polynomial (listed in order) divide the real number line into intervals in which the polynomial has no sign changes. These zeros are called the **critical numbers** of the inequality and the intervals are called the **test intervals** of the inequality.

Example: $x^2+2x-3=(x+3)(x-1)=0$
x = -3, 1 → Critical Values
$(-\infty,-3),(-3,1),(1,\infty)$ → Test Intervals

To solve: Test one value from each interval.

Interval	x-value	Value of the polynomial	Sign of the polynomial
$(-\infty,-3)$	-4	$(-4)^2+2(-4)-3$	Positive
$(-3,1)$	0	$(0)^2+2(0)-3$	Negative
$(1,\infty)$	2	$(2)^2+2(2)-3$	Positive

The graph verifies this result:



Example: Solve: $2x^2 - 5x > 12$

Solution:

$$2x^2 - 5x > 12$$
$$2x^2 - 5x - 12 > 0$$
$$(2x+3)(x-4) > 0$$

→ *Critical Numbers:* $x = -\frac{3}{2}$ and $x = 4$

→ *Test Intervals:* $\left(-\infty, -\frac{3}{2}\right), \left(-\frac{3}{2}, 4\right), (4, \infty)$

→ *Test:* Is $(2x+3)(x-4) > 0$?

After testing these intervals, you can see that the polynomial $2x^2 - 5x - 12$ is positive in the open intervals $\left(-\infty, -\frac{3}{2}\right)$ and $(4, \infty)$. Therefore, the solution set of the inequality is $\left(-\infty, -\frac{3}{2}\right) \cup (4, \infty)$.

Unusual Solution Sets

- **A polynomial with no real solutions.** Since this type of polynomial has no zeros, it will be entirely positive or entirely negative.

- **A polynomial with one real solution.** Since this type of polynomial touches the *x*-axis but does not cross it, it will be entirely positive or entirely negative excluding the *x*-value for which its value is zero.

Rational Inequalities

Critical numbers and test intervals are useful in solving inequalities involving rational expressions. However, the critical numbers come from two places in the expression.

Critical Numbers:

1) Zeros of the numerator (*x*-values for which the value of the expression is zero)
2) Zeros of the denominator (*x*-values for which the expression is undefined)

Example: Solve $\frac{3x-12}{x-5} \leq 4$

Solution: First, you must simplify the inequality so that one side is zero.

$$\frac{3x-12}{x-5} - 4 \leq 0$$

$$\frac{3x-12}{x-5} - \frac{4(x-5)}{x-5} \leq 0$$

→ *Critical Numbers:* $x = 8$ and $x = 5$

$$\frac{3x-12-4x+20}{x-5} \leq 0$$

→ *Test Intervals:* $(-\infty, 5), (5, 8), (8, \infty)$

$$\frac{-x+8}{x-5} \leq 0$$

→ *Test:* Is $\frac{-x+8}{x-5} \leq 0$?

Testing these intervals, you can see that the expression (-x + 8)/(x – 5) is negative in the open intervals $(-\infty,5)$ and $(8,\infty)$. In addition, the expression equals zero at $x = 8$. Therefore, the solution set of the inequality is $(-\infty,5)\cup[8,\infty)$.

Finding the Domain of an Expression

Recall that the domain of an expression is the set of all x-values for which the expression is defined. Inequalities can help you determine this domain.

Example: Find the domain of $\sqrt{100-25x^2}$.

Solution: Since you can only find the square root of a positive number, the expression $100 - 25x^2$ must be nonnegative. Thus, you can solve the following inequality to find the domain:

$$144-16x^2 \geq 0$$

$$9-x^2 \geq 0$$

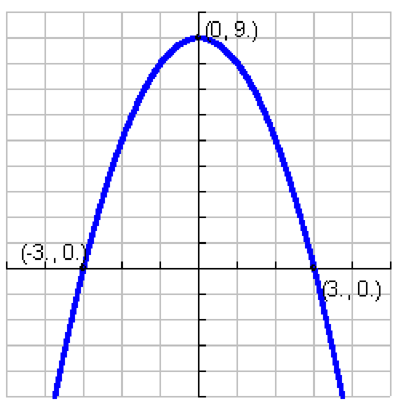
$$(3-x)(3+x) \geq 0$$

→ *Critical Numbers:* $x = 3$ and $x = -3$

→ *Test Intervals:* $(-\infty,-3),(-3,3),(3,\infty)$

→ *Test:* Is $(3-x)(3+x) \geq 0$?

The solution set for this inequality is the *closed interval* $[-3,3]$. The graph below confirms this solution set.



Vertical Motion Model

The vertical motion of an object can be modeled by the equation

$$s = -16t^2 + v_o t + s_o$$

where s is the height in feet of the object, v_o is the initial velocity of the object, s_o is the initial height of the object, and t is time in seconds.

Example: A model rocket is fired straight upward from ground level with an initial velocity of 200 feet per second. During what time period will its height *exceed* 500 feet?

Solution: Substituting these values into the vertical motion model, we obtain

$$500 < -16t^2 + 200t.$$

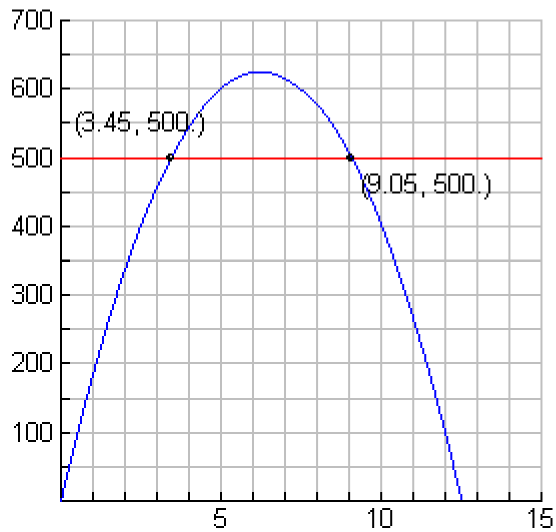
Solving this algebraically requires the quadratic formula. Setting the inequality equal to zero and then using the quadratic formula yields two critical numbers, at $t \approx 3.45$ seconds and $t \approx 9.05$ seconds. Therefore, the rocket exceeds 500 feet between 3.45 seconds and 9.05 seconds after being launched.

Graphically, you could use the INTERSECT feature of the calculator to quickly see the solution set. In this case, you would enter the following into the Y= screen:

$$\begin{cases} y_1 = -16x^2 + 200x \\ y_2 = 500 \end{cases}$$

Next, you need to set the appropriate window. The graph shown below uses the following values:

$$\begin{cases} X\min = 0 \\ X\max = 15 \\ X\text{scl} = 1 \\ Y\min = 0 \\ Y\max = 700 \\ Y\text{scl} = 50 \end{cases}$$



Using 2nd CALC → Intersect you can find the two points of intersection. The *x*-values correspond to the *t*-values.

Practice Exercises

Solve the inequality and sketch the solution on the real number line.

1. $3(x-4) < 5x + 10$

2. $-10 \le 2 - 3(x+1) < 12$

3. $|6x| - 8 \le 16$

4. $4\left|\frac{1}{2}x - 6\right| + 6 > 18$

5. $(x+4)^2 > 25$

6. $3x^3 + 15x^2 \le 0$

7. $\dfrac{x+4}{x-1} + 3 < 0$

8. $\dfrac{x+1}{x^2-16} \ge \dfrac{1}{x+4}$

10. Find the domain of $\sqrt{9x^2-16}$

11. Find the domain of $\sqrt{x^2+6x-40}$

12. Alex throws a tennis ball upward from height of 4.5 feet with an initial velocity of 8 feet per second. If he hits the ball when it returns to a height of 4 feet, how long is the ball in the air?